

# The Range of Approximate Unitary Equivalence Classes of Homomorphisms from AH-algebras

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## Abstract

Let  $C$  be a unital AH-algebra and  $A$  be a unital simple  $C^*$ -algebra with tracial rank zero. It has been shown that two unital monomorphisms  $\phi, \psi : C \rightarrow A$  are approximately unitarily equivalent if and only if

$$[\phi] = [\psi] \text{ in } KL(C, A) \text{ and } \tau \circ \phi = \tau \circ \psi \text{ for all } \tau \in T(A),$$

where  $T(A)$  is the tracial state space of  $A$ . In this paper we prove the following: Given  $\kappa \in KL(C, A)$  with  $\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$  and with  $\kappa([1_C]) = [1_A]$  and a continuous affine map  $\lambda : T(A) \rightarrow T_{\mathbf{f}}(C)$  which is compatible with  $\kappa$ , where  $T_{\mathbf{f}}(C)$  is the convex set of all faithful tracial states, there exists a unital monomorphism  $\phi : C \rightarrow A$  such that

$$[\phi] = \kappa \text{ and } \tau \circ \phi(c) = \lambda(\tau)(c)$$

for all  $c \in C_{s.a.}$  and  $\tau \in T(A)$ . Denote by  $\text{Mon}_{au}^e(C, A)$  the set of approximate unitary equivalence classes of unital monomorphisms. We provide a bijective map

$$\Lambda : \text{Mon}_{au}^e(C, A) \rightarrow KLT(C, A)^{++},$$

where  $KLT(C, A)^{++}$  is the set of compatible pairs of elements in  $KL(C, A)^{++}$  and continuous affine maps from  $T(A)$  to  $T_{\mathbf{f}}(C)$ .

Moreover, we realized that there are compact metric spaces  $X$ , unital simple AF-algebras  $A$  and  $\kappa \in KL(C(X), A)$  with  $\kappa(K_0(C(X))_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$  for which there is no homomorphism  $h : C(X) \rightarrow A$  so that  $[h] = \kappa$ .

## 1 Introduction

Recall that an AH-algebra is a  $C^*$ -algebra which is an inductive limit of  $C^*$ -algebras  $C_n$ , where  $C_n = P_n M_{r(n)}(C(X_n)) P_n$  for some finite CW complex  $X_n$  and projections  $P_n \in M_{r(n)}(C(X_n))$ . Note that every unital separable commutative  $C^*$ -algebra is an AH-algebra and every AF-algebra is an AH-algebra. It was shown in [11] (see also Theorem 3.6 of [13]) that two unital monomorphisms  $\phi, \psi : C \rightarrow A$ , where  $A$  is a unital simple  $C^*$ -algebra with tracial rank zero, are approximately unitarily equivalent if and only if

$$[\phi] = [\psi] \text{ and } \tau \circ \phi(c) = \tau \circ \psi(c)$$

for all  $c \in C_{s.a.}$  and  $\tau \in T(A)$ . This result plays a role in the study of classification of amenable  $C^*$ -algebras, or otherwise known as the Elliott program. It also has applications in the study of dynamical systems both classical and non-commutative ones (see [11]). It is desirable to know the range of the approximately unitary equivalence classes of monomorphisms from a unital AH-algebra  $C$  into a unital simple  $C^*$ -algebra with tracial rank zero. For example, one may ask if given any  $\kappa \in KL(C, A)$  and any continuous affine map  $\lambda : T(A) \rightarrow T(C)$  there exists a monomorphism  $\phi$  such that  $[\phi] = \kappa$  and  $\tau \circ h(c) = \lambda(\tau)(c)$  for all  $c \in C_{s.a.}$  and  $\tau \in T(A)$ .

When  $C$  is a finite CW complex, it was shown (see also a previous result of L. Li [6]) in [10] that, for any  $\kappa \in KK(C, A)$  with  $\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$  and with  $\kappa([1_C]) = [1_A]$ , there exists a unital monomorphism  $\phi : C \rightarrow A$  such that  $[\phi] = \kappa$ . It should be noted that both conditions that  $\kappa([1_C]) = [1_A]$  and  $\kappa(K_0(C) \setminus \{0\}) \subset K_0(A) \setminus \{0\}$  are necessary for the existence of such  $\phi$ . One of the earliest such results (concerning monomorphisms from  $C(\mathbb{T}^2)$  into a unital simple AF-algebra) of this kind appeared in a paper of Elliott and Loring ([3] see also [2]). It was shown in [10] that the same result holds for the case that  $C$  is a unital simple AH-algebra which has real rank zero, stable rank one and weakly unperforated  $K_0(C)$ . Therefore, it is natural to expect that it holds for general unital AH-algebras.

Let  $C$  be the unitization of  $\mathcal{K}$ , the algebra of compact operators on  $l^2$ . Then it does not have a faithful tracial state. Consequently, it can not be embedded into any unital UHF-algebra, or any unital simple  $C^*$ -algebra which has at least one tracial state (It has been shown that a unital AH-algebra  $C$  can be embedded into a unital simple AF-algebra if and only if  $C$  admits a faithful tracial state—see [13]). This example at least suggests that for general unital AH-algebras, the problem is slightly more complicated than the first thought. Moreover, we note that to provide the range of approximately unitary equivalence classes of unital monomorphisms from  $C$ , we also need to consider the map  $\lambda : T(A) \rightarrow T(C)$ . Let  $X$  be a compact metric space and let  $C = C(X)$ . Suppose that  $h : C \rightarrow A$  is a unital monomorphism and suppose that  $\tau \in T(A)$ . Then  $\tau \circ h$  induces a Borel probability measure on  $X$ . Suppose that  $\kappa \in KL(C, A)$  is given. It is clear that not every measure  $\mu$  can be induced by those  $h$  for which  $[h] = \kappa$ . Thus, we should consider a compatible pair  $(\kappa, \gamma)$  which gives a more complete information on  $K$ -theory than either  $\kappa$  or  $\gamma$  alone.

The main result of this paper is to show that if  $C$  is a unital AH-algebra,  $A$  is any unital simple  $C^*$ -algebra with tracial rank zero,  $\kappa \in KL(C, A)^{++}$  (see 2.3 below) with  $\kappa([1_C]) = [1_A]$  and  $\lambda : T(A) \rightarrow T_{\mathbf{f}}(C)$ , where  $T_{\mathbf{f}}(C)$  is the convex set of faithful tracial states, which is a continuous affine map and is compatible with  $\kappa$ , there is indeed a unital monomorphism  $\phi : C \rightarrow A$  such that

$$[\phi] = \kappa \text{ in } KL(C, A) \text{ and } \phi_T = \lambda.$$

We also show that the existence of  $\lambda$  is essential to provide homomorphisms  $\phi$ . In fact, we find out that there are compact metric spaces  $X$ , unital simple AF-algebras  $A$  and  $\kappa \in KL(C(X), A)^{++}$  with  $\kappa([1_C]) = [1_A]$  for which there are no  $\lambda : T(A) \rightarrow T_{\mathbf{f}}(C(X))$  which is compatible with  $\kappa$ . Moreover, we discovered that there are no homomorphism  $h : C \rightarrow A$  (not just monomorphisms) such that  $[h] = \kappa$ . This further demonstrates that tracial information is an integral part of  $K$ -theoretical information.

## 2 Notation

**2.1.** Let  $A$  be a unital  $C^*$ -algebra. Denote by  $T(A)$  the tracial state space of  $A$ . Denote by  $\text{Aff}(T(A))$  the space of all real affine continuous functions on  $T(A)$ . If  $\tau \in T(A)$ , we will also use  $\tau$  for the tracial state  $\tau \otimes \text{Tr}$  on  $M_k(A)$  for all integer  $k \geq 1$ , where  $\text{Tr}$  is the standard trace on  $M_k$ . If  $a \in A_{s.a.}$ , denote by  $\check{a}$  a real affine function in  $\text{Aff}(T(A))$  defined by  $\check{a}(\tau) = \tau(a)$  for all  $\tau \in T(A)$ .

Let  $C$  be another unital  $C^*$ -algebra. Suppose that  $\gamma : \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$  is a positive linear map. We say it is unital if  $\gamma(1_C)(\tau) = 1$ . We say it is strictly positive, if  $a \in \text{Aff}(T(A))_+ \setminus \{0\}$ , then  $\gamma(a)(\tau) > 0$  for all  $\tau \in T(A)$ .

Suppose that  $\phi : C \rightarrow A$  is a unital homomorphism. Denote by  $h_T : T(A) \rightarrow T(C)$  the affine continuous map induced by  $h$ , i.e.,

$$h_T(\tau)(c) = \tau \circ h(c) \text{ for all } c \in C.$$

It also induces a positive linear map  $h_{\sharp} : \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$  defined by

$$h_{\sharp}(\check{a})(\tau) = \tau \circ h(a) \text{ for all } a \in C_{s,a} \text{ and } \tau \in T(A),$$

where  $\check{a}(\tau) = \tau(a)$  for  $\tau \in T(A)$ .

If  $\lambda : T(A) \rightarrow T(C)$  is an affine continuous map, then it gives a unital positive linear map  $\lambda_{\sharp} : \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$  by

$$\lambda_{\sharp}(f)(\tau) = f(\lambda(\tau)) \text{ for all } f \in \text{Aff}(T(C)) \text{ and for all } \tau \in T(A).$$

Conversely, a unital positive linear map  $\gamma : \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$  gives an affine continuous map  $\gamma_T : T(A) \rightarrow T(C)$  by

$$f(\gamma_T(\tau)) = \gamma(f)(\tau) \text{ for all } f \in \text{Aff}(T(C)) \text{ and } \tau \in T(C).$$

Suppose that  $A$  is a unital simple  $C^*$ -algebra. Then  $\gamma$  is strictly positive if and only if  $\gamma_T$  maps  $T(A)$  into  $T_{\text{f}}(C)$ .

Denote by  $\rho_A : K_0(A) \rightarrow \text{Aff}(T(A))$  the positive homomorphism induced by  $\rho_A([p])(\tau) = \tau(p)$  for all projections  $p \in M_{\infty}(A)$  and  $\tau \in T(A)$ .

Let  $A$  and  $C$  be two unital  $C^*$ -algebras and let  $\kappa_0 : K_0(C) \rightarrow K_0(A)$  be a unital positive homomorphism ( $\kappa_0([1_C]) = [1_A]$ ). Suppose that  $\lambda : T(A) \rightarrow T(C)$  is a continuous affine map. We say that  $\lambda$  is compatible with  $\kappa_0$ , if  $\tau(\kappa([p])) = \lambda(\tau)(p)$  for all projections  $p$  in  $M_{\infty}(A)$ . Similarly, a unital positive linear map  $\gamma : \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$  is said to be compatible with  $\kappa_0$ , if  $\gamma(\check{p})(\tau) = \tau(\kappa([p]))$  for all projections  $p$  in  $M_{\infty}(C)$ .  $\gamma$  is compatible with  $\kappa_0$  if and only if  $\gamma_T$  is so.

Two projections in  $A$  are equivalent if there exists a partial isometry  $w \in A$  such that  $w^*w = p$  and  $ww^* = q$ .

**2.2.** Let  $A$  be a unital  $C^*$ -algebra and let  $C$  be a separable  $C^*$ -algebra which satisfies the universal coefficient theorem. By a result of Dadarlat and Loring ([1]),

$$KL(C, A) = \text{Hom}_{\Lambda}(\underline{K}(C), \underline{K}(A)), \quad (\text{e 2.1})$$

where, for any  $C^*$ -algebra  $B$ ,

$$\underline{K}(B) = \bigoplus_{i=0,1} K_i(B) \bigoplus_{n=2}^{\infty} \bigoplus_{i=0,1} K_i(B, \mathbb{Z}/n\mathbb{Z}).$$

We will identify two objects in (e 2.1). Denote by

$$\underline{K}_{F,k}(C) = \bigoplus_{i=0,1} K_i(B) \bigoplus_{n|k} \bigoplus_{i=0,1} K_i(B, \mathbb{Z}/n\mathbb{Z}).$$

If  $K_i(C)$  is finitely generated ( $i = 0, 1$ ), then there is  $k_0 \geq 1$  such that

$$\text{Hom}_{\Lambda}(\underline{K}(C), \underline{K}(A)) \cong \text{Hom}_{\Lambda}(F_{k_0}\underline{K}(C), F_{k_0}\underline{K}(A))$$

(see [1]).

**Definition 2.3.** Denote by  $KL(C, A)^{++}$  the set of those  $\kappa \in \text{Hom}_{\Lambda}(\underline{K}(C), \underline{K}(A))$  such that

$$\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A) \setminus \{0\}.$$

Denote by  $KL_e(C, A)^{++}$  the set of those  $\kappa \in KL(C, A)^{++}$  such that  $\kappa([1_C]) = [1_A]$ .

**Definition 2.4.** Let  $\kappa \in KL_e(C, A)^{++}$  and let  $\lambda : T(A) \rightarrow T(C)$  be a continuous affine map. We say that  $\lambda$  is compatible with  $\kappa$  if  $\lambda$  is compatible with  $\kappa|_{K_0(C)}$ . Let  $\gamma : \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$  be a positive linear map. We say  $\gamma$  is compatible with  $\kappa$  if  $\gamma$  is compatible with  $\kappa|_{K_0(C)}$ , i.e.,  $\tau \circ \kappa([p]) = \gamma(\tilde{p})(\tau)$  for all projections  $p \in M_\infty(C)$ .

**2.5.** Let  $C = C(X)$  for some compact metric space  $X$ . One has the following short exact sequence:

$$0 \rightarrow \ker \rho_C \rightarrow K_0(C) \rightarrow C(X, \mathbb{Z}) \rightarrow 0.$$

It is then easy to see that, for every projection  $p \in M_\infty(C)$ , there is a projection  $q \in C$  and an integer  $n$  such that  $\rho_A([p]) = n\rho_A([q])$ . It follows that if  $C$  is a unital AH-algebra, then for every projection  $p \in M_\infty(C)$ , there is a projection  $q \in C$  and an integer  $n \geq 1$  such that

$$\rho_A([p]) = n\rho_A([q]).$$

Note also that in this case  $\text{Aff}(T(C)) = C_{s.a.}$ . Therefore, in this note, instead of considering a unital positive linear maps  $\gamma : \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$ , we may consider unital positive linear maps  $\gamma : C_{s.a.} \rightarrow \text{Aff}(T(A))$ . Moreover,  $\gamma$  is compatible with some  $\kappa \in KL(C, A)^{++}$ , if  $\gamma(p)(\tau) = \tau(\kappa([p]))$  for all projections  $p \in C$  and  $\tau \in T(A)$ .

**2.6.** Let  $\phi, \psi : C \rightarrow A$  be two maps between  $C^*$ -algebras. Let  $\epsilon > 0$  and  $\mathcal{F} \subset C$  be a subset. We write

$$\phi \approx_\epsilon \psi \text{ on } \mathcal{F},$$

if

$$\|\phi(c) - \psi(c)\| < \epsilon \text{ for all } c \in \mathcal{F}.$$

**2.7.** Let  $L : C \rightarrow A$  be a linear map. Let  $\delta > 0$  and  $\mathcal{G} \subset C$  be a (finite) subset. We say  $L$  is  $\delta$ - $\mathcal{G}$ -multiplicative if

$$\|L(ab) - L(a)L(b)\| < \delta \text{ for all } a, b \in \mathcal{G}.$$

**Definition 2.8.** Let  $A$  be a unital  $C^*$ -algebra. Denote by  $U(A)$  the unitary group of  $A$ . Let  $B \subset A$  be another  $C^*$ -algebra and  $\phi : B \rightarrow A$  be a map. We write  $\phi = \text{ad } u$  for some  $u \in U(A)$  if  $\phi(b) = u^*bu$  for all  $b \in B$ .

Let  $\phi, \psi : C \rightarrow A$  be two maps. We say that  $\phi$  and  $\psi$  are approximately unitarily equivalent if there exists a sequence of unitaries  $\{u_n\} \subset A$  such that

$$\lim_{n \rightarrow \infty} \text{ad } u_n \circ \phi(c) = \psi(c) \text{ for all } c \in C.$$

### 3 Approximate unitary equivalence

We begin with the following theorem

**Theorem 3.1.** (Theorem 3.6 of [13] and see also Theorem 3.4 of [11]) *Let  $C$  be a unital AH-algebra and let  $A$  be a unital simple  $C^*$ -algebra with tracial rank zero. Suppose that  $\phi, \psi : C \rightarrow A$  are two unital monomorphisms. Then there exists a sequence of unitaries  $\{u_n\} \subset A$  such that*

$$\lim_{n \rightarrow \infty} \text{ad } u_n \circ \psi(c) = \phi(c) \text{ for all } c \in C,$$

*if and only if*

$$[\phi] = [\psi] \text{ in } KL(C, A) \text{ and } \tau \circ \phi = \tau \circ \psi \text{ for all } \tau \in T(A).$$

We need the following variation of results in [11].

**Theorem 3.2.** *Let  $C$  be a unital AH-algebra, let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) = 0$  and let  $\gamma : C_{s.a.} \rightarrow \text{Aff}(T(A))$  be a unital strictly positive linear map.*

*For any  $\epsilon > 0$  and any finite subset  $\mathcal{F} \subset C$ , there exist  $\eta > 0$ ,  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C$ , a finite subset  $\mathcal{H} \subset C_{s.a.}$  and a finite subset  $\mathcal{P} \subset \underline{K}(C)$  satisfying the following:*

*Suppose that  $L_1, L_2 : C \rightarrow A$  are two unital completely positive linear maps which are  $\delta$ - $\mathcal{G}$ -multiplicative such that*

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}, \quad (\text{e3.2})$$

$$|\tau \circ L_i(g) - \gamma(g)(\tau)| < \eta \text{ for all } g \in \mathcal{H}, i = 1, 2. \quad (\text{e3.3})$$

*Then there is a unitary  $u \in A$  such that*

$$\text{ad } u \circ L_2 \approx_{\epsilon} L_2 \text{ on } \mathcal{F}. \quad (\text{e3.4})$$

*Proof.* Write  $C = \overline{\bigcup_{n=1}^{\infty} C_n}$ , where  $C_n = P_n M_{r(n)}(C(X_n)) P_n$ , where  $X$  is a compact subset of a finite CW complex and where  $P_n \in M_{r(n)}(C(X_n))$  is a projection. Let  $\epsilon > 0$  and a finite subset  $\mathcal{F} \subset C$  be fixed. Without loss of generality, we may assume that  $\mathcal{F} \subset C_1$ . Let  $\eta_0 > 0$  such that

$$|f(x) - f(x')| < \epsilon/8 \text{ for all } f \in \mathcal{F},$$

if  $\text{dist}(x, x') < \eta_0$ . Let  $\{x_1, x_2, \dots, x_m\} \subset X$  be  $\eta_0/2$ -dense in  $X$ . Suppose that  $O_i \cap O_j = \emptyset$  if  $i \neq j$ , where

$$O_j = \{x \in X : \text{dist}(x, x_j) < \eta_0/2s\}, \quad j = 1, 2, \dots, m$$

for some integer  $s \geq 1$ .

Choose non-zero element  $g_j \in (C_1)_{s.a.}$  such that  $0 \leq g_j \leq 1$  whose support lies in  $O_j$ ,  $j = 1, 2, \dots, m$ . Note such  $g_j$  exists (by taking those in the center for example). Choose

$$\sigma_0 = \min\{\inf\{\gamma(g_j)(\tau) : \tau \in T(A)\} : 1 \leq j \leq m\}.$$

Since  $\gamma$  is strictly positive,  $\sigma_0 > 0$ . Set  $\sigma = \min\{\sigma_0/2, 1/2s\}$ . Then, by Corollary 4.8 of [11], such  $\delta > 0$ ,  $\eta > 0$ ,  $\mathcal{G}$ ,  $\mathcal{H}$  and  $\mathcal{P}$  exists. □

**Lemma 3.3.** *Let  $X$  be a compact metric space, let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) = 0$  and let  $\gamma : C(X)_{s.a.} \rightarrow \text{Aff}(T(A))$  be a unital strictly positive linear map.*

*Then, for any  $\epsilon > 0$  and any  $\mathcal{F} \subset C(X)$ , there exists  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)_{s.a.}$ , a set  $S_1, S_2, \dots, S_n$  of mutually disjoint clopen subsets with  $\bigcup_{i=1}^n S_i = X$ , satisfying the following:*

*For any two unital homomorphisms  $\phi_1, \phi_2 : C(X) \rightarrow pAp$  with finite dimensional range for some projection  $p \in A$  with  $\tau(1 - p) < \delta$  such that*

$$[\phi_1(\chi_{S_i})] = [\phi_2(\chi_{S_i})] \text{ in } K_0(A), \quad i = 1, 2, \dots, n, \quad (\text{e3.5})$$

$$|\tau \circ \phi_1(g) - \gamma(g)(\tau)| < \delta \text{ and} \quad (\text{e3.6})$$

$$|\tau \circ \phi_2(g) - \gamma(g)(\tau)| < \delta \quad (\text{e3.7})$$

*for all  $g \in \mathcal{G}$  and for all  $\tau \in T(A)$ , there exist a unitary  $u \in U(pAp)$  such that*

$$\text{ad } u \circ \phi_1 \approx_{\epsilon} \phi_2 \text{ on } \mathcal{F}. \quad (\text{e3.8})$$

*Proof.* This follows from 3.2 immediately. There is a sequence of finite CW complex  $X_n$  such that  $C(X) = \lim_{n \rightarrow \infty} (C(X_n), h_n)$ , where each  $h_n$  is a unital homomorphism. Fix  $\epsilon > 0$  and a finite subset  $\mathcal{F} \subset C(X)$ . Without loss of generality, we may assume that  $\mathcal{F} \subset h_K(\mathcal{F}_K)$  for some integer  $K \geq 1$  and a finite subset  $\mathcal{F}_K$ .

Given any finite subset  $\mathcal{P} \subset \underline{K}(C(X))$ , one obtains a finite subset  $\mathcal{Q}_k \subset \underline{K}(C(X_k))$  such that  $[h_k](\mathcal{Q}_k) = \mathcal{P}$  for some  $k \geq 1$ . Let  $p_1, p_2, \dots, p_n$  be mutually orthogonal projections corresponding to the connected components of  $X_k$ . To simplify notation, without loss of generality, we may assume that  $k = K$ .

There are mutually disjoint clopen sets  $S_1, S_2, \dots, S_n$  of  $X$  with  $\cup_{i=1}^n S_i = X$  such that  $h_k(p_i) = \chi_{S_i}$ ,  $i = 1, 2, \dots, n$ . Since  $\phi$  and  $\psi$  are homomorphisms with finite dimensional range, if

$$[\phi(\chi_{S_i})] = [\psi(\chi_{S_i})] \text{ in } K_0(A),$$

then

$$[\phi \circ h_k] = [\psi \circ h_k] \text{ in } KL(C(X_k), A).$$

This, in particular, implies that

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}.$$

This above argument shows that the lemma follows from 3.2. □

**Definition 3.4.** Let  $X$  be a compact metric space which is a compact subset of some finite CW complex  $Y$ . Then there exists a decreasing sequence of finite CW complexes  $X_n \subset Y$  such that

$$X \subset X_n \text{ and } \lim_{n \rightarrow \infty} \text{dist}(X_n, X) = 0.$$

Denote by  $s_{m,n} : C(X_m) \rightarrow C(X_n)$  (for  $n > m$ ) and  $s_n : C(X_n) \rightarrow C(X)$  be the surjective homomorphisms induced by the inclusion  $X_{n+1} \subset X_n$  and  $X \subset X_n$ , respectively.

**Lemma 3.5.** *Let  $Y$  be a finite CW complex and  $X \subset Y$  be a compact subset. For any  $\epsilon > 0$ , any finite subset  $\mathcal{F} \subset C(X)$ , there exists a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$ , an integer  $k \geq 1$  and an integer  $N \geq 1$  satisfying the following:*

*For any unital homomorphisms  $\phi, \psi : C(X_m) \rightarrow A$  ( $m \geq k$ ) for any unital simple  $C^*$ -algebra with  $TR(A) = 0$  for which*

$$[\phi]|_{\mathcal{Q}} = [\psi]|_{\mathcal{Q}} \text{ in } KL(C(X_m), A),$$

*where  $\mathcal{Q} \subset \underline{K}(C(X_m))$  is a finite subset such that  $[s_m](\mathcal{Q}) = \mathcal{P}$ , then there exists a unitary  $U \in M_{N+1}(A)$  such that*

$$adU \circ (\phi \oplus \Phi \circ s_m) \approx_{\epsilon} (\psi \oplus \Phi \circ s_m) \text{ on } s_m^{-1}(\mathcal{F}),$$

*where  $\Phi : C(X) \rightarrow M_N(A)$  is defined by*

$$\Phi(f) = \text{diag}(f(x_1), f(x_2), \dots, f(x_N)) \text{ for all } f \in C(X), \quad (\text{e3.9})$$

*where  $\{x_1, x_2, \dots, x_N\}$  is a finite subset of  $X$ .*

*Proof.* Assume that the lemma were false. Then there would be a positive number  $\epsilon_0 > 0$ , a finite subset  $\mathcal{F}_0 \subset C(X)$ , an increasing sequence of finite subsets  $\{\mathcal{P}_n\} \subset \underline{K}(C(X))$  with  $\cup_n \mathcal{P}_n = \underline{K}(C(X))$ , a sequence of unital  $C^*$ -algebras, two subsequences  $\{R(n)\}, \{k(n)\}$  of  $\mathbb{N}$  and two sequences monomorphisms  $\phi_n, \psi_n : C(X_{k(n)}) \rightarrow A_n$  such that

$$[\phi_n]|_{\mathcal{Q}_n} = [\psi_n]|_{\mathcal{Q}_n} \text{ in } KK(C(X_{k(n)}), A_n) \text{ and} \quad (\text{e3.10})$$

$$\limsup_n \{ \inf \{ \max \{ \|u_n^*(\phi_n \oplus \Phi_n \circ s_n)(f)u_n - (\phi \oplus \Phi_n \circ s_n)(f)\| : f \in s_m^{-1}(\mathcal{F}) \} \} \} \geq \epsilon_0, \quad (\text{e3.11})$$

where infimum is taken among all possible  $\Phi_n : C(X) \rightarrow M_{R(n)}(A_n)$  with the form described above and among all possible unitaries  $\{u_n\} \subset U(M_{R(n)+1}(A))$ , and where  $\mathcal{Q}_n \subset \underline{K}(C(X_{k(n)}))$

is a finite subset such that  $[s_{k(n)}](\mathcal{Q}_n) = \mathcal{P}_n$ . Since  $K_i(C(X_n))$  is finitely generated, by passing to a subsequence, if necessary, without loss of generality, we may assume (see also the end of 2.2) that

$$[\phi_{n+1} \circ s_{k(n),k(n+1)}] = [\psi_{n+1} \circ s_{k(n),k(n+1)}] \text{ in } KL(C(X_{k(n)}), A), \quad n = 1, 2, \dots \quad (\text{e 3.12})$$

Let  $\phi_n^{(m)} = \phi_m$ , if  $n \leq m$ ,  $\phi_n^{(m)} = \phi_n \circ s_{m,n}$ ,  $\psi_n^{(m)} = \psi_m$ , if  $n \leq m$  and  $\psi_n^{(m)} = \psi_n \circ s_{m,n}$ ,  $n = 1, 2, \dots$ . Denote by  $H_1^{(m)}, H_2^{(m)} : C(X_{k(m)}) \rightarrow \prod_n A_n$  by  $H_1^{(m)}(f) = \{\phi_n^{(m)}\}$  and  $H_2^{(m)}(f) = \{\psi_n^{(m)}\}$ . Let  $\pi : \prod_n A_n \rightarrow \prod_n A_n / \bigoplus_n A_n$  be the quotient map. Then  $\pi \circ H_1^{(m)}$  and  $\pi \circ H_2^{(m)}$  both have spectrum  $X$ . Moreover, for each  $i$ , all  $\pi \circ H_i^{(m)}$  gives the same homomorphism  $F_i : C(X) \rightarrow \prod_n A_n / \bigoplus_n A_n$ ,  $i = 1, 2$ .

Since  $TR(A_n) = 0$ ,  $A_n$  has real rank zero, stable rank one, weakly unperforated  $K_0(A_n)$ , by Corollary 2.1 of [5] and (e 3.12)

$$[H_1^{(m+1)} \circ s_{k(m),k(m+1)}] = [H_2^{(m+1)} \circ s_{k(m),k(m+1)}] \text{ in } KL(C(X_{k(m)}), \prod_n A_n)$$

It follows from Corollary 2.1 of [5] again that

$$[F_1] = [F_2] \text{ in } KL(C, \prod_n A_n / \bigoplus_n A_n).$$

It then follows from Theorem 1.1 and the Remark 1.1 of [5] that there is an integer  $N \geq 1$  and a unitary  $W \in U(M_{N+1}(\prod_n A_n / \bigoplus_n A_n))$  such that

$$\text{ad } W \circ (F_2 \oplus H_0) \approx_{\epsilon_0/2} (F_1 \oplus H_0) \text{ on } \mathcal{F}_0, \quad (\text{e 3.13})$$

where  $H_0 : C(X) \rightarrow M_N(\prod_n A_n / \bigoplus_n A_n)$  is defined by  $H_0(f) = \sum_{i=1}^N f(x_i) E_i$  for all  $f \in C(X)$ ,

$x_i \in X$  and  $E_i = \text{diag}(\overbrace{0, \dots, 0}^{i-1}, 1, 0, \dots, 0)$ ,  $i = 1, 2, \dots, N$ .

There is a unitary  $\{W_n\} \in U(\prod_n A_n)$  such that  $\pi(\{W_n\}) = W$ . Then, for some sufficiently large  $n$ ,

$$W_n^* \text{diag}(\phi_n(f), f(x_1), f(x_2), \dots, f(x_N)) W_n \approx_{\epsilon_0} (\psi_n(f), f(x_1), f(x_2), \dots, f(x_N)) \quad (\text{e 3.14})$$

on  $\mathcal{F}_0$ . This contradicts (e 3.11). □

**Remark 3.6.** There exists a positive number  $\eta > 0$  and integer  $N_1 > 0$  which depend only on  $\epsilon$  and  $\mathcal{F}$  such that  $\{x_1, x_2, \dots, x_N\}$  and an integer  $N$  can be replaced by any  $\eta$ -dense finite subset  $\{\xi_1, \xi_2, \dots, \xi_{N_1}\}$  and integer  $N_1$ .

From the proof, we also know that the assumption that  $A$  has tracial rank zero can be replaced by much weaker conditions (see Corollary 2.1 of [5]). The main difference of 3.5 and results in [5] is that homomorphisms  $\phi$  and  $\psi$  are not assumed to be from  $C(X)$  to  $A$ .

## 4 Monomorphisms from $C(X)$

**Lemma 4.1.** *Let  $X$  be a finite CW complex and let  $A$  be a unital simple  $C^*$ -algebra with real rank zero, stable rank one and weakly unperforated  $K_0(A)$ . Let  $e_1, e_2, \dots, e_m \in C(X)$  be mutually orthogonal projections corresponding to connected components of  $X$ .*

*Suppose that  $\kappa \in KK(C(X), A)^{++}$  with  $\kappa([1_{C(X)}]) = [1_A]$ . Then, for any projection  $p \in A$  and any unital homomorphism  $\phi_0 : C(X) \rightarrow (1-p)A(1-p)$  with finite dimensional range such*

that  $\phi_0([e_i]) < \kappa([e_i])$ ,  $i = 1, 2, \dots, m$ . Then there exists a unital monomorphism  $\phi_1 : C(X) \rightarrow pAp$  such that

$$[\phi_1 + \phi_0] = \kappa \text{ in } KK(C(X), A). \quad (\text{e4.15})$$

*Proof.* Since  $\sum_{i=1}^m \kappa([e_i]) = [1_A]$  and  $A$  has stable rank one, there are mutually orthogonal projections  $p_1, p_2, \dots, p_m \in A$  such that

$$\sum_{i=1}^m p_i = 1_A \text{ and } [p_i] = \kappa([e_i]), \quad i = 1, 2, \dots, m \quad (\text{e4.16})$$

From this it is clear that we may reduce the general case to the case that  $X$  is connected.

So now we assume that  $X$  is connected. Then it is easy to see that

$$\kappa - [\phi_0] \in KK(C(X), A)^{++}$$

and  $(\kappa - [\phi_0])([1_{C(X)}]) = p$ . It follows from Theorem 4.7 of [10] that there is a monomorphism  $\phi_1 : C(X) \rightarrow pAp$  such that

$$[\phi] = \kappa - [\phi_0].$$

□

**Lemma 4.2.** *Let  $X$  a compact metric space and let  $A$  be a unital simple  $C^*$ -algebra with tracial rank zero. Suppose that  $\gamma : C(X)_{s,a} \rightarrow \text{Aff}(T(A))$  is a unital strictly positive linear map. Let  $S_1, S_2, \dots, S_n$  be a set of mutually disjoint clopen subsets of  $X$  with  $\cup_{i=1}^n S_i = X$ . Then for any  $\delta > 0$  and any finite subset  $\mathcal{G} \subset C(X)_{s,a}$ , there exists a projection  $p \in A$  with  $p \neq 1_A$  and a unital homomorphism  $h : C(X) \rightarrow pAp$  with finite dimensional range such that*

$$|\tau \circ h(g) - \gamma(g)(\tau)| < \delta \text{ for all } g \in \mathcal{G} \text{ and } \tau \in T(A), \text{ and} \quad (\text{e4.17})$$

$$\tau \circ h(\chi_{S_i}) < \gamma(\chi_{S_i})(\tau) \text{ for all } \tau \in T(A), \quad (\text{e4.18})$$

$i = 1, 2, \dots, n$ .

*Proof.* Put

$$d = \min\{\delta, \min\{\inf\{\gamma(\chi_{S_i})(\tau) : \tau \in T(A)\} : 1 \leq i \leq n\}\}.$$

Since  $\gamma$  is strictly positive,  $d > 0$ .

Let  $\mathcal{G}_0 = \mathcal{G} \cup \{\chi_{S_1}, \chi_{S_2}, \dots, \chi_{S_n}\}$ . It follows from 4.3 of [12] that there is a unital homomorphism  $h_0 : C(X) \rightarrow A$  with finite dimensional range such that

$$|\tau \circ h(g) - \gamma(g)(\tau)| < d/8n \text{ for all } g \in \mathcal{G}_0 \quad (\text{e4.19})$$

and for all  $\tau \in T(A)$ . In particular,

$$|\tau \circ h(\chi_{S_i}) - \gamma(\chi_{S_i})(\tau)| < d/8n \text{ for all } \tau \in T(A) \quad (\text{e4.20})$$

$i = 1, 2, \dots, n$ .

Since  $\rho_A(K_0(A))$  is dense in  $\text{Aff}(T(A))$ , there exists a projection  $p_0 \in A$  such that

$$d/2n < \tau(p_0) < d/n \text{ for all } \tau \in T(A). \quad (\text{e4.21})$$

Note that  $\tau(p_0) < \gamma(\chi_{S_i})(\tau)$  for all  $\tau \in T(A)$ ,  $i = 1, 2, \dots, n$ . Moreover, by (e4.20),

$$\tau \circ h(\chi_{S_i}) > \gamma(\chi_{S_i})(\tau) - d/8n \geq d - d/8n > \tau(p_0). \quad (\text{e4.22})$$



for all  $\tau \in T(A)$ .

Write  $h_0(f) = \sum_{k=1}^m f(x_k)e_k$  for all  $f \in C(X)$ , where  $x_k \in X$  and  $\{e_1, e_2, \dots, e_m\}$  is a set of mutually orthogonal projections with  $\sum_{k=1}^m e_k = 1_A$ .

Note that

$$h_0(\chi_{S_j}) = \sum_{x_k \in S_j} e_k.$$

Therefore (by (e 4.22))

$$[p_0] \leq \left[ \sum_{x_k \in S_j} e_k \right]. \quad (\text{e 4.23})$$

By Zhang's Riesz interpolation property (see [14]), there are projections  $e'_k \leq e_k$  such that

$$[p_0] = \left[ \sum_{k \in S_j} e'_k \right].$$

By Zhang's half projection theorem (see Theorem 1.1 of [15]), for each  $k$ , there is a projection  $e''_k \leq e'_k$  such that

$$[e''_k] + [e''_k] \geq [e'_k]. \quad (\text{e 4.24})$$

Thus

$$2 \left[ \sum_{\chi_k \in S_i} e''_k \right] \geq [p_0], \quad i = 1, 2, \dots, n. \quad (\text{e 4.25})$$

Therefore (by (e 4.21) and (e 4.20))

$$\tau \left( \sum_{x_k \in S_i} (e_k - e''_k) \right) < \tau \circ h_0(\chi_{S_i}) - (1/2)\tau(p_0) \quad (\text{e 4.26})$$

$$< \tau \circ h(\chi_{S_i}) - d/4n \quad (\text{e 4.27})$$

$$< \gamma(\chi_{S_i})(\tau) - d/8n \text{ for all } \tau \in T(A). \quad (\text{e 4.28})$$

Let  $p = \sum_{k=1}^m (e_k - e''_k)$ . Then clearly that  $p \neq 1$ . Moreover,

$$\tau(1 - p) < d/4 \text{ for all } \tau \in T(A).$$

Define  $h(f) = \sum_{k=1}^m f(x_k)(e_k - e''_k)$  for all  $f \in C(X)$ . Then

$$|\tau \circ h(f) - \tau \circ h_0(f)| < \tau \left( \sum_{k=1}^m e''_k \right) = \tau(1 - p) < d/4 < \delta \quad (\text{e 4.29})$$

for all  $\tau \in T(A)$ .

Then, by (e 4.28),

$$\tau \circ h(\chi_{S_i}) < \gamma(\chi_{S_i})(\tau) \text{ for all } \tau \in T(A). \quad (\text{e 4.30})$$

□

**Lemma 4.3.** *Let  $X$  a compact metric space and let  $A$  be a unital simple  $C^*$ -algebra with tracial rank zero. Suppose that  $\gamma : C(X)_{s,a} \rightarrow \text{Aff}(T(A))$  is a unital strictly positive linear map. Let  $S_1, S_2, \dots, S_n$  be a set of mutually disjoint clopen subsets of  $X$  with  $\cup_{i=1}^n S_i = X$ . Then for any  $\delta > 0$ ,  $\eta > 0$ , for any integer  $N$  and any  $\eta$ -dense subset  $\{x_1, x_2, \dots, x_N\}$  of  $X$  and any finite*

subset  $\mathcal{G} \subset C(X)_{s,a}$ , there exists a projection  $p \in A$  with  $p \neq 1_A$  and a unital homomorphism  $h : C(X) \rightarrow pAp$  with finite dimensional range such that

$$|\tau \circ h(g) - \gamma(g)(\tau)| < \delta \text{ for all } g \in \mathcal{G} \text{ and } \tau \in T(A), \text{ and} \quad (\text{e 4.31})$$

$$\tau \circ h(\chi_{S_i}) < \gamma(\chi_{S_i})(\tau) \text{ for all } \tau \in T(A), \quad (\text{e 4.32})$$

$i = 1, 2, \dots, n$ ,

$$h(f) = \sum_{i=1}^N f(x_i) e_i \oplus h_1(f) \text{ for all } f \in C(X), \quad (\text{e 4.33})$$

where  $h_1 : C(X) \rightarrow (1 - \sum_{i=1}^N e_i)A(1 - \sum_{i=1}^N e_i)$  is a unital homomorphism with finite dimensional range and  $\{e_1, e_2, \dots, e_N\}$  is a set of mutually orthogonal projections such that  $[e_i] = [e_1] \geq [1 - p]$ ,  $i = 1, 2, \dots, N$ .

*Proof.* Let  $N \geq 1$  and let  $\eta$ -dense subset  $\{x_1, x_2, \dots, x_N\}$  of  $X$  be given. Let  $\eta_0 > 0$  such that

$$|f(x) - f(x')| < \delta/4 \text{ for all } f \in \mathcal{G}, \quad (\text{e 4.34})$$

provided that  $\text{dist}(x, x') < \eta_0$ .

Choose  $\eta_0 > \eta_1 > 0$  such that  $B(x_i, \eta_1)$  intersects with one and only one  $S_i$  among  $\{S_1, S_2, \dots, S_n\}$ .

Choose, for each  $i$ , a non-zero function  $f_i \in C(X)$  with  $0 \leq f \leq 1$  whose support is in  $B(x_i, \eta_1/2)$ . Put

$$d_0 = \min\{\inf\{\gamma(f_i)(\tau) : \tau \in T(A)\} : 1 \leq i \leq N\}.$$

So  $d_0 > 0$ . Put  $\delta_1 = \min\{\delta/8, \delta_0/4\}$  and put  $\mathcal{G}_1 = \mathcal{G} \cup \{1_{C(X)}\} \cup \{f_i : i = 1, 2, \dots, N\}$ .

Now applying 4.2. We obtain a projection  $p \in A$  and a unital homomorphism  $h_0 : C(X) \rightarrow pAp$  such that

$$|\tau \circ h_0(g) - \gamma(g)(\tau)| < \delta_1 \text{ for all } g \in \mathcal{G}_1 \text{ and} \quad (\text{e 4.35})$$

$$\tau \circ h_0(\chi_{S_i}) < \gamma(\chi_{S_i})(\tau) \quad (\text{e 4.36})$$

for all  $\tau \in T(A)$ ,  $i = 1, 2, \dots, n$ . Since  $1_{C(X)} \in \mathcal{G}_1$ , by (e 4.35),

$$\tau(1 - p) < \delta_1 < \delta_0/4 \text{ for all } \tau \in T(A). \quad (\text{e 4.37})$$

Write  $h_0(f) = \sum_{j=1}^L f(\xi_j) q_j$  for all  $f \in C(X)$ , where  $\xi_j \in X$  and  $\{q_1, q_2, \dots, q_L\}$  is a set of mutually orthogonal projections with  $\sum_{j=1}^L q_j = p$ .

Define

$$e'_i = \sum_{\xi_j \in B(x_i, \eta_1/2)} q_j, \quad i = 1, 2, \dots, N.$$

It follows from (e 4.35) that, for each  $i$ ,

$$\tau(e'_i) \geq \tau \circ h_0(f_i) \quad (\text{e 4.38})$$

$$> \gamma(f_i)(\tau) - \delta_1 > 3\delta_0/4 \geq \tau(p) \quad (\text{e 4.39})$$

for all  $\tau \in T(A)$ . It follows that

$$[e'_i] \geq [p], \quad i = 1, 2, \dots, N.$$

There are projections  $e_i \leq e'_i$  such that

$$[e_i] = [e_1] \geq [1 - p], \quad i = 1, 2, \dots, N. \quad (\text{e 4.40})$$

Define

$$h_1(f) = \sum_{\xi_j \notin \bigcup_{i=1}^N B(x_i, \eta_1/2)} f(\xi_j) q_j + \sum_{i=j}^N f(x_j)(e'_i - e_i) \quad \text{and} \quad (\text{e 4.41})$$

$$h(f) = \sum_{i=1}^N f(x_i) e_i \oplus h_1(f) \quad (\text{e 4.42})$$

for all  $f \in C(X)$ . Since  $B(x_j, \eta_1/2)$  lies in one of  $S_i$ ,

$$\tau \circ h(\chi_{S_i}) = \tau \circ h_0(\chi_{S_i}) \quad \text{for all } \tau \in T(A),$$

$i = 1, 2, \dots, n$ . It follows from (e 4.36) that (e 4.32) holds. By the choice of  $\eta_0$ , we also have

$$\|h_0(g) - h(g)\| < \delta/2 \quad \text{for all } h \in \mathcal{G}. \quad (\text{e 4.43})$$

Thus, by (e 4.35), (e 4.31) also holds. □

**Lemma 4.4.** *Let  $X$  be a compact metric space and let  $A$  be a unital simple  $C^*$ -algebra with tracial rank zero. Suppose that  $\gamma : C(X)_{s.a} \rightarrow \text{Aff}(T(A))$  is a unital strictly positive linear map which is compatible with a strictly positive homomorphism  $\kappa_0 : K_0(C(X)) \rightarrow K_0(A)$ . Fix  $\delta > 0$ ,  $\eta > 0$ , a finite subset  $\mathcal{F} \subset C(X)_{s.a.}$ , an integer  $N \geq 1$ , an  $\eta$ -dense subset  $\{x_1, x_2, \dots, x_N\}$  of  $X$ , a finitely many mutually disjoint clopen subset  $S_1, S_2, \dots, S_n \subset X$  with  $\bigcup_{i=1}^n S_i = X$ , a finite subset set  $\{a_1, a_2, \dots, a_n\} \subset A$  of mutually orthogonal projections with*

$$0 < a_i < \kappa_0([\chi_{S_i}]), \quad i = 1, 2, \dots, n,$$

*a finitely many mutually disjoint clopen subsets  $\{F_1, F_2, \dots, F_{n_1}\}$  of  $X$  with  $\bigcup_{i=1}^{n_1} F_i = X$ , and a projection  $p$  with  $\tau(p) = \tau(\sum_{i=1}^n a_i)$  for all  $\tau \in T(A)$ .*

*There is a projection  $q \in A$  such that  $[p] \leq [q]$  and a unital homomorphism  $h : C(X) \rightarrow qAq$  with finite dimensional range such that*

$$|\tau \circ h(g) - \gamma(g)(\tau)| < \delta \quad \text{for all } g \in \mathcal{F} \text{ and } \tau \in T(A), \quad \text{and} \quad (\text{e 4.44})$$

$$\tau \circ h(\chi_{F_i}) < \gamma(\chi_{F_i})(\tau) \quad \text{for all } \tau \in T(A), \quad (\text{e 4.45})$$

$i = 1, 2, \dots, n$ ,

$$h(f) = \sum_{i=1}^N f(x_i) e_i \oplus h_1(f) \quad \text{for all } f \in C(X), \quad (\text{e 4.46})$$

*where  $h_1 : C(X) \rightarrow (1 - \sum_{i=1}^N e_i)A(1 - \sum_{i=1}^N e_i)$  is a unital homomorphism with finite dimensional range and  $\{e_1, e_2, \dots, e_N\}$  is a set of mutually orthogonal projections such that  $[e_i] = [e_1] \geq [1 - p]$ ,  $i = 1, 2, \dots, N$ .*

*Moreover, there exists a projection  $p' \in q$  such that*

$$p'h(f) = h(f)p' \quad \text{for all } f \in C(X) \quad \text{and} \quad (\text{e 4.47})$$

$$[h(\chi_{S_j})p'] = [a_j], \quad j = 1, 2, \dots, n. \quad (\text{e 4.48})$$

*Proof.* Let

$$d_0 = \min\{\inf\{\tau(\kappa_0([\chi_{S_i}])) - [a_i] : \tau \in T(A)\} : 1 \leq i \leq n\}$$

and let

$$d_1 = \inf\{\tau(1 - p) : \tau \in T(A)\}.$$

Then  $d_0, d_1 > 0$ . Define  $\delta_1 = \min\{\delta/4, d_0/2, d_1/2\}$  and  $\mathcal{G}_1 = \mathcal{F} \cup \{1_{C(X)}, \chi_{S_i}, i = 1, 2, \dots, n\}$ . By applying 4.3, we obtain a projection  $q \in A$  and a unital homomorphism  $h : C(X) \rightarrow qAq$  with finite dimensional range satisfying the following:

$$|\tau \circ h(g) - \gamma(g)(\tau)| < \delta_1 \text{ for all } g \in \mathcal{G}_1, \quad (\text{e 4.49})$$

$$\tau \circ h(\chi_{F_j}) < \gamma(\chi_{F_j})(\tau) \quad (\text{e 4.50})$$

for all  $\tau \in T(A)$ ,  $j = 1, 2, \dots, n_1$ ,

$$h(f) = \sum_{k=1}^N f(x_k) e_i \oplus h_1(f) \text{ for all } f \in C(X), \quad (\text{e 4.51})$$

where  $\{e_1, e_2, \dots, e_N\}$  is a set of mutually orthogonal and mutually equivalent projections such that  $[e_1] \geq [1 - q]$ , and where  $h_1 : C(X) \rightarrow (q - \sum_{k=1}^N e_k)A(q - \sum_{k=1}^N e_k)$  is a unital homomorphism with finite dimensional range.

Since  $1_{C(X)} \in \mathcal{G}_1$ , by the choice of  $\delta_1$ , we conclude that  $[p] \leq [q]$ .

Moreover, by (e 4.49),

$$\tau \circ h(\chi_{S_i}) > \tau(a_i) \text{ for all } \tau \in T(A), \quad i = 1, 2, \dots, n. \quad (\text{e 4.52})$$

Write

$$h(f) = \sum_{s=1}^L f(\xi_s) E_s \text{ for all } f \in C(X),$$

where  $\xi_s \in X$  and  $\{E_1, E_2, \dots, E_L\}$  is a set of mutually orthogonal projections such that  $\sum_{s=1}^L E_s = q$ . By (e 4.52), one has

$$\sum_{\xi_s \in S_i} E_s \geq a_i, \quad i = 1, 2, \dots, n. \quad (\text{e 4.53})$$

For each  $i$ , by the Riesz Interpolation Property ([14]), there is a projection  $E'_s \leq E_s$  for which  $x_s \in S_i$  such that

$$[\sum_{\xi_s \in S_i} E'_s] = [a_i]. \quad (\text{e 4.54})$$

Put  $p' = \sum_{s=1}^L E'_s$  Then

$$p'h(f) = h(f)p' \text{ for all } f \in C(X) \text{ and} \quad (\text{e 4.55})$$

$$[h(\chi_{S_i})p'] = [a_i], \quad i = 1, 2, \dots, n. \quad (\text{e 4.56})$$

□

**Theorem 4.5.** *Let  $X$  be a compact subset of a finite CW complex and let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) = 0$ . Suppose that  $\kappa \in KL_e(C(X), A)^{++}$  and suppose that there exists a unital strictly positive linear map  $\gamma : C(X)_{s.a} \rightarrow \text{Aff}(T(A))$  which is compatible with  $\kappa$ . Then there exists a unital monomorphism  $\phi : C(X) \rightarrow A$  such that*

$$[\phi] = \kappa \text{ in } KL(C, A).$$

*Proof.* Suppose that  $X \subset Y$ , where  $Y$  is a finite CW complex. Let  $X_n \subset Y$  be a decreasing sequence of finite CW complexes for which 3.4 holds. Suppose that  $p_{n,1}, p_{n,2}, \dots, p_{n,r(n)}$  are mutually orthogonal projections of  $C(X_n)$  which correspond to the connected components of  $X_n$ . It is clear that we may assume that each connected component of  $X_n$  contains at least one point of  $X$ . This implies that  $[s_n] \in KK(C(X_n), C(X))^{++}$ . It follows that

$$[s_n] \times \kappa \in KK(C(X_n), A)^{++}. \quad (\text{e 4.57})$$

Let  $\{\mathcal{F}_n\}$  be an increasing sequence of finite subsets of  $C(X)$  whose union is dense in  $C(X)$ . Let  $\{\eta_n\}$  be a decreasing sequence of positive numbers with  $\lim_{n \rightarrow \infty} \eta_n = 0$ ,  $\{\mathcal{P}_n\}$  be an increasing sequence of finite subsets of  $\underline{K}(C)$  whose union is  $\underline{K}(C)$ , let  $\{k(n)\}, \{N(n)\} \subset \mathbb{N}$  be two sequences of integers such that  $k(n), N(n) \nearrow \infty$ , and  $\{x_{n,1}, x_{n,2}, \dots, x_{n,N(n)}\}$  be  $\eta_n$ -dense subsets of  $X$  which satisfy the requirements of 3.5 and 3.6 for corresponding  $\epsilon_n = 1/2^{n+2}$  and  $\mathcal{F}_n$ .

By passing to a subsequence if necessary, we may assume that there is a finite subset  $\mathcal{F}'_n \subset C(X_{k(n+1)})$  such that  $s_{k(n+1)}(\mathcal{F}'_n) = \mathcal{F}_n$  and a finite subset  $\mathcal{Q}_{k(n)} \subset \underline{K}(C(X_{k(n)}))$  such that  $[s_{k(n)}](\mathcal{Q}_{k(n)}) = \mathcal{P}_n$ ,  $n = 1, 2, \dots$ . We may assume that  $1_{C(X_{k(n)})} \in \mathcal{F}'_n$ , without loss of generality.

Set  $\kappa_n = [s_{k(n)}] \times \kappa$ . Note that  $\kappa_n([1_{C(X_{k(n)})}]) = [1_A]$ .

Let  $\delta_n$  (in place of  $\delta$ ),  $\mathcal{G}'_n \subset C(X)_{s.a.}$  (in place of  $\mathcal{G}$ ),  $S_{1,n}, S_{2,n}, \dots, S_{m(n),n}$  (in place of  $\{S_1, S_2, \dots\}$ ) be a set of disjoint clopen subsets of  $X$  with  $\cup_{i=1}^{m(n)} S_i = X$  required by 3.3 for  $\epsilon_n$  and  $\mathcal{F}_n$ ,  $n = 1, 2, \dots$ . We may assume that  $1_{C(X)} \in \mathcal{G}'_n$ ,  $n = 1, 2, \dots$ .

By taking a refinement of the clopen partition of  $X$ , we may assume that  $s_n(p_{n,i})$  is a finite sum of functions in  $\{\chi_{S_{j,n}} : 1 \leq j \leq m(n)\}$ ,  $i = 1, 2, \dots, r(n)$ .

Let  $\mathcal{G}_n \subset C(X_{k(n)})_{s.a.}$  be a finite subsets for which  $s_{k(n)}(\mathcal{G}_n) = \mathcal{G}'_n$ ,  $n = 1, 2, \dots$ .

By applying 4.3, we obtain a projection  $P_1 \in A$  and a unital homomorphism  $\Phi'_1 : C(X) \rightarrow P_1 A P_1$  such that

$$|\tau \circ \Phi'_1(g) - \gamma(g)(\tau)| < \delta_1/2 \text{ for all } g \in \mathcal{G}'_1, \quad (\text{e 4.58})$$

$$\tau \circ \Phi'_1(\chi_{S_{j,1}}) < \gamma(\chi_{S_{j,1}})(\tau) \quad (\text{e 4.59})$$

for all  $\tau \in T(A)$ ,  $i = 1, 2, \dots, m(1)$ , and

$$\Phi'_1(f) = \sum_{i=1}^{N(1)} f(x_{1,i}) e_i^{(1)} \oplus \Phi'_{0,1}(f) \text{ for all } f \in C(X), \quad (\text{e 4.60})$$

where  $\{e_1^{(1)}, e_2^{(1)}, \dots, e_{N(1)}^{(1)}\}$  is a set of mutually orthogonal and mutually equivalent projections with  $[e_1] \geq [(1 - P_1)]$  and where  $\Phi'_{0,1} : C(X) \rightarrow (P_1 - \sum_{i=1}^{N(1)} e_i^{(1)}) A ((P_1 - \sum_{i=1}^{N(1)} e_i^{(1)}))$  is a unital homomorphism with finite dimensional range. Note also, since  $1_{C(X)} \in \mathcal{G}'_1$ ,  $\tau(1 - P_1) < \delta_1/2$  for all  $\tau \in T(A)$ .

It follows from 4.1 that there is a unital monomorphism  $\phi'_1 : C(X_{k(1)}) \rightarrow (1 - P_1) A (1 - P_1)$  such that

$$[\phi'_1] + [\Phi'_1 \circ s_1] = \kappa_1 \text{ in } KK(C(X_{k(1)}), A). \quad (\text{e 4.61})$$

Define  $\phi_1 = \phi'_1 + \Phi'_1 \circ s_1$ .

Suppose that, for  $1 \leq m \leq n$ , there are unital homomorphisms  $\phi'_m : C(X_{k(m)}) \rightarrow (1 - P_m) A (1 - P_m)$  and  $\Phi'_m : C(X) \rightarrow P_m A P_m$  and a unital (injective) homomorphism  $\phi_m = \phi'_m + \Phi'_m \circ s_{k(m)}$  such that

- (1) there are mutually orthogonal and mutually equivalent projections  $e_1^{(m)}, e_2^{(m)}, \dots, e_{N(m)}^{(m)} \in P_m A P_m$  for which  $[e_1^{(m)}] \geq [1 - P_m]$ , and

$$\Phi'_m(f) = \sum_{i=1}^{N(m)} f(x_{m,i}) e_i^{(m)} \oplus \Phi_m^{(0)}(f) \text{ for all } f \in C(X)$$

where  $\Phi_m^{(0)} : C(X) \rightarrow (P_m - \sum_{i=1}^{N(m)} e_i^{(m)}) A (P_m - \sum_{i=1}^{N(m)} e_i^{(m)})$  is a unital homomorphism with finite dimensional range;

- (2)  $\tau \circ \Phi'_m(\chi_{S_{j,m}}) < \gamma(\chi_{S_{j,m}})(\tau)$  for all  $\tau \in T(A)$ ,  $j = 1, 2, \dots, m(m)$ ;  
(3)  $|\tau \circ \Phi'_m(g) - \gamma(g)(\tau)| < \delta_m/2$  for all  $g \in \mathcal{G}'_m$  and for all  $\tau \in T(A)$ ;  
(4)  $[P_{m+1}] \geq [P_m]$  in  $K_0(A)$  and  $\tau(1 - P_m) < \delta_m/2$  for all  $\tau \in T(A)$ ;  
(5) there is a projection  $P'_{m+1} \leq P_{m+1}$  such that  $P'_{m+1} \Phi_{m+1} = \Phi'_{m+1} P_{m+1}$  and

$$[\Phi'_{m+1}(\chi_{S_{j,m}}) P'_{m+1}] = [\Phi'_m(\chi_{S_{j,m}})] \text{ in } K_0(A), j = 1, 2, \dots, m(m);$$

- (6)  $\phi'_m$  is a unital monomorphism;  
(7)  $[\phi_m] = [\phi'_m] + [\Phi'_m \circ s_{k(m)}] = \kappa_m$ ;  
(8) there exists a unitary  $u_m \in A$  such that

$$\text{ad } u_m \circ \phi_{m+1} \circ s_{k(m), k(m+1)} \approx_{1/2^{m+1}} \phi_m \text{ on } s_{k(m)}^{-1}(\mathcal{F}_m), \quad m = 1, 2, \dots, n-1.$$

It follows from 4.4 that there is a projection  $P_{n+1} \in A$  and a unital homomorphism  $\Phi'_{n+1} : C(X) \rightarrow P_{n+1} A P_{n+1}$  satisfying the following:

- (1) there are mutually orthogonal and mutually equivalent projections  $e_1^{(n+1)}, e_2^{(n+1)}, \dots, e_{N(n+1)}^{(n+1)} \in P_{n+1} A P_{n+1}$  for which  $[e_1^{(n+1)}] \geq [1 - P_{n+1}]$ , and

$$\Phi'_{n+1}(f) = \sum_{i=1}^{N(n+1)} f(x_{n+1,i}) e_i^{(n+1)} \oplus \Phi_{n+1}^{(0)}(f) \text{ for all } f \in C(X)$$

where  $\Phi_{n+1}^{(0)} : C(X) \rightarrow (P_{n+1} - \sum_{i=1}^{N(n+1)} e_i^{(n+1)}) A (P_{n+1} - \sum_{i=1}^{N(n+1)} e_i^{(n+1)})$  is a unital homomorphism with finite dimensional range;

- (2)  $\tau \circ \Phi'_{n+1}(\chi_{S_{j,n+1}}) < \gamma(\chi_{S_{j,n+1}})(\tau)$  for all  $\tau \in T(A)$ ,  $j = 1, 2, \dots, m(n+1)$ ;  
(3)  $|\tau \circ \Phi'_{n+1}(g) - \gamma(g)(\tau)| < \delta_{n+1}/2$  for all  $g \in \mathcal{G}'_{n+1}$  and for all  $\tau \in T(A)$ ;  
(4)  $[P_{n+1}] \geq [P_n]$  in  $K_0(A)$  and  $\tau(1 - P_{n+1}) < \delta_{n+1}/2$  for all  $\tau \in T(A)$ ;  
(5) there is a projection  $P'_{n+1} \leq P_{n+1}$  such that  $P'_{n+1} \Phi'_{n+1} = \Phi'_{n+1} P_{n+1}$  and

$$[\Phi'_{n+1}(\chi_{S_{n,j}}) P'_{n+1}] = [\Phi'_n(\chi_{S_{n,j}})] \text{ in } K_0(A), j = 1, 2, \dots, m(n).$$

It follows from 4.1 that there is a unital monomorphism  $\phi'_{n+1} : C(X_{k(n+1)}) \rightarrow (1 - P_{n+1})A(1 - P_{n+1})$  such that

$$[\phi'_{n+1}] = \kappa_{n+1} - [\Phi'_{n+1} \circ s_{k(n+1)}] \text{ in } KK(C(X_{k(n+1)}), A) \quad (\text{e 4.62})$$

Define  $\phi_{n+1} = \phi'_{n+1} + \Phi'_{n+1} \circ s_{k(n+1)}$ .

Thus  $\phi'_{n+1}$ ,  $\phi'_{n+1}$  and  $\phi_{n+1}$  satisfy (1), (2), (3), (4), (5), (6) and (7).

To complete the induction, define  $\Phi''_{n+1} : C(X) \rightarrow P'_{n+1}AP'_{n+1}$  by  $\Phi''_{n+1}(f) = P'_{n+1}\Phi'_{n+1}(f)P'_{n+1}$  for all  $f \in C(X)$ . By (3) and (4),

$$|\tau \circ \Phi''_{n+1}(g) - \gamma(g)(\tau)| < \delta_{n+1}/2 \text{ for all } g \in \mathcal{G}_n$$

for all  $\tau \in T(A)$ . Note that, by (5),  $[P'_{n+1}] = [P_n]$ . There is a unitary  $w_n \in U(A)$  such that

$$w_n^* P'_{n+1} w_n = P_n.$$

Thus, by (5) and (3), and by applying 3.3, there exists a unitary  $v_n \in U(P_n A P_n)$  such that

$$\text{ad } v_n \circ \text{ad } w_n \circ \Phi''_{n+1} \approx_{\epsilon_n} \Phi'_n \text{ on } \mathcal{F}_n. \quad (\text{e 4.63})$$

Denote  $\Psi'_{n+1} = P'_{n+1}\Phi_{n+1}P'_{n+1}$  and  $\Psi_{n+1} = \text{ad } w_n \circ \Psi'_{n+1}$ . Let  $\phi''_{n+1} = \text{ad } w_n \circ \phi'_{n+1} \oplus \Psi_{n+1}$ . Now consider  $\phi'_n$  and  $\phi''_{n+1} \circ s_{k(n),k(n+1)}$ . By (7) and (e 4.62), we have

$$[\phi''_{n+1} \circ s_{k(n),k(n+1)}]|_{\mathcal{Q}_{k(n)}} = [\phi'_n]|_{\mathcal{Q}_{k(n)}}.$$

It follows from (1) and 3.5 that there exists a unitary  $V_n \in U(A)$  such that

$$\text{ad } V_n \circ (\phi''_{n+1} \circ s_{k(n),k(n+1)} \oplus \Phi'_n \circ s_{k(n)}) \approx_{\epsilon_n} \phi'_n \oplus \Phi'_n \circ s_{k(n)} \text{ on } s_{k(n)}^{-1}(\mathcal{F}_n). \quad (\text{e 4.64})$$

Define  $u_n = w_n(v_n + (1 - P_n))V_n$ . Then, by (e 4.63) and (e 4.64),

$$\text{ad } u_n \circ \phi_{n+1} \approx_{2\epsilon_n} \phi_n \text{ on } s_{k(n)}^{-1}(\mathcal{F}_n). \quad (\text{e 4.65})$$

Note  $2\epsilon_n = 1/2^{n+1}$ .

This concludes the induction.

Define  $\psi_1 = \phi_1$  and  $\psi_{n+1} = \text{ad } u_n \circ \phi_{n+1}$ ,  $n = 1, 2, \dots$ . Then, by (8) above,

$$\|\psi_n(c) - \psi_{n+1} \circ s_{k(n),k(n+1)}(c)\| < 1/2^{n+2} \text{ for all } c \in s_{k(n)}^{-1}(\mathcal{F}_n), \quad (\text{e 4.66})$$

$n = 1, 2, \dots$

Fix  $m$  and  $f \in \mathcal{F}_m$ , let  $g \in s_{k(m)}^{-1}(\mathcal{F}_m)$  such that  $s_{k(m)}(g) = f$ .

It follows that  $\{\psi_n \circ s_{m,n}(g)\}_{n \geq m}$  is a Cauchy sequence by (e 4.66).

Note that if  $g' \in s_{k(m)}^{-1}(\mathcal{F}_m)$  such that  $s_{k(m)}(g) = s_{k(m)}(g')$ , then, for any  $\epsilon > 0$ , there exists  $n \geq m$  such that

$$\|s_{k(m),k(n)}(g) - s_{k(m),k(n)}(g')\| < \epsilon.$$

Thus  $h(f) = \lim_{n \rightarrow \infty} \psi_n \circ s_{m,n}(g)$  is well-defined. It is then easy to verify that  $h$  defines a unital homomorphism from  $C(X)$  into  $A$ . Since each  $\phi_n$  is injective, it is easy to check that  $h$  is also injective.

If  $x \in \mathcal{Q}_m$ , then by (7) above,

$$[h] \circ [s_{k(m)}](x) = \kappa_n \circ [s_{k(m),k(n)}](x) = \kappa \circ [s_{k(m)}](x).$$

Therefore

$$[h] = \kappa \text{ in } KL(C, A).$$

It is also easy to check from (3) and (4) that

$$\tau \circ h(g) = \gamma(\check{g})(\tau) \text{ for all } g \in C(X)_{s,a} \quad (\text{e 4.67})$$

and for all  $\tau \in T(A)$ .

□

## 5 AH-algebras

**Lemma 5.1.** *Let  $X$  be a compact subset of a finite CW complex, let  $C = PM_k(C(X))P$ , where  $P \in M_k(C(X))$  is a projection, and let  $A$  be a unital simple  $C^*$ -algebra with tracial rank zero. Suppose that  $\kappa \in KL_e(C, A)^{++}$  and suppose that  $\gamma : C_{s.a.} \rightarrow \text{Aff}(T(A))$  is a unital positive linear map which is compatible with  $\kappa$ . Then, for any  $\epsilon > 0$  and finite subset  $\mathcal{F} \subset C(X)$ , there is a unital monomorphism  $\phi : C \rightarrow A$  such that*

$$[\phi] = \kappa \text{ and } \tau \circ \phi(f) = \gamma(f)(\tau) \text{ for all } \tau \in T(A). \quad (\text{e 5.68})$$

*Proof.* It is clear that the case that  $C = M_k(C(X))$  follows from 4.5 immediately. For the general case, there is an integer  $d \geq 1$  and a projection  $p \in M_d(C)$  such that  $pM_d(C)p \cong M_m(C(X))$  for some integer  $m \geq 1$ . Thus the general case is reduced to the case that  $C = M_m(C(X))$ .  $\square$

**Theorem 5.2.** *Let  $C$  be a unital AH-algebra and let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) = 0$ . Suppose that  $\kappa \in KL_e(C, A)^{++}$ . Suppose also that there is a unital strictly positive linear map  $\gamma : C_{s.a.} \rightarrow \text{Aff}(T(A))$  which is compatible with  $\kappa$ . Then there is a monomorphism  $\phi : C \rightarrow A$  such that*

$$[\phi] = \kappa \text{ in } KL(C, A) \text{ and} \quad (\text{e 5.69})$$

$$\tau \circ \phi(c) = \gamma(c)(\tau) \quad (\text{e 5.70})$$

for all  $c \in C_{s.a.}$  and  $\tau \in T(A)$ .

*Proof.* We may write  $C = \overline{\bigcup_{n=1}^{\infty} C_n}$ , where  $C_n = P_n M_k(C(X_n)) P_n$ , where  $X_n$  is a compact subset of a finite CW complex and  $P_n \in M_k(C(X_n))$  is a projection. We may also assume that  $1_{C_n} = 1_C$  for all  $n$ . Denote by  $\iota_n : C_n \rightarrow C$  the embedding,  $n = 1, 2, \dots$

Define

$$\kappa_n = \kappa \circ [\iota_n] \text{ and } \gamma_n = \gamma \circ (\iota_n)_{\sharp}$$

$n = 1, 2, \dots$  Since  $\iota_n$  is injective  $\kappa_n \in KL_e(C_n, A)^{++}$  and  $\gamma_n$  is unital strictly positive. It is also clear that  $\gamma_n$  is compatible with  $\kappa_n$ , since  $\gamma$  is compatible with  $\kappa$ . It follows from 5.1 that there is a sequence of unital monomorphisms  $\phi_n : C_n \rightarrow A$  such that

$$[\phi_n] = \kappa_n \text{ and } \tau \circ \phi_n(c) = \gamma_n(c)(\tau) \quad (\text{e 5.71})$$

for all  $c \in C_{s.a.}$  and  $\tau \in T(A)$ .

Let  $\{\mathcal{F}_n\}$  be an increasing sequence of finite subsets of  $C$  whose union is dense in  $C$ . By passing to a subsequence, if necessary, without loss of generality, we may assume that  $\mathcal{F}_n \subset C_n$ .

It follows (from 2.3.13 of [9], for example) that there is, for each  $n$ , a unital completely positive linear map  $L_n : C \rightarrow A$  such that

$$L_n \approx_{1/2^{n+1}} \phi_n \circ \iota_n \text{ on } \mathcal{F}_n. \quad (\text{e 5.72})$$

It follows from Lemma 5.1, by passing to a subsequence again and by applying (e 5.71), there is a sequence of unitaries  $u_n$  and a subsequence of  $\{k(n)\}$  such that

$$\text{ad } u_n \circ L_{k(n+1)} \approx_{1/2^n} L_{k(n)} \text{ on } \mathcal{F}_n, \quad (\text{e 5.73})$$

$n = 1, 2, \dots$  Define  $\psi_1 = L_1$ ,  $\psi_{n+1} = \text{ad } u_n \circ L_{k(n+1)}$ ,  $n = 1, 2, \dots$ . Note that  $\{\psi_n(c)\}$  is a Cauchy sequence in  $A$  for each  $c \in \mathcal{F}_m$ . Define  $h(c) = \lim_{n \rightarrow \infty} \psi_n(c)$ . It is easy to see that  $h$  gives a unital homomorphism from  $C$  into  $A$ . Moreover, for each  $x \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$ ,

$$h(x) = \lim_{n \rightarrow \infty} \text{ad } u_n \circ \phi_{k(n)} \circ \iota_{k(n)} \circ \dots \circ \iota_n(x). \quad (\text{e 5.74})$$



Since each  $\phi_n$  is injective, it follows that  $h$  is a monomorphism. From (e 5.74) and (e 5.71), we have

$$[h] = \kappa \text{ as well as } \tau \circ h(c) = \gamma(c)(\tau)$$

for all  $c \in C_{s.a.}$  and  $\tau \in T(A)$ .

□

**Corollary 5.3.** *Let  $X$  be a compact metric space and let  $A$  be a unital simple  $C^*$ -algebra with tracial rank zero. Suppose that  $\kappa \in \text{KL}_e(C(X), A)^{++}$ . Suppose also that there is a unital strictly positive linear map  $\gamma : C_{s.a.} \rightarrow \text{Aff}(T(A))$  which is compatible with  $\kappa$ . Then there is a monomorphism  $\alpha : C \rightarrow A$  such that*

$$[\alpha] = \kappa \text{ in } \text{KL}(C(X), A) \text{ and} \quad (\text{e 5.75})$$

$$\tau \circ \phi(c) = \gamma(c)(\tau) \quad (\text{e 5.76})$$

for all  $c \in C(X)_{s.a.}$  and  $\tau \in T(A)$ .

**Example 5.4.** Let  $X = \{\frac{-1}{n} : n \in \mathbb{N}\} \cup [0, 1] \cup \{1 + \frac{1}{n} : n \in \mathbb{N}\} \subset [-1, 2]$ . Put  $C = C(X)$ . Then

$$K_0(C(X)) = C(X, \mathbb{Z}).$$

Take two sequences of positive rational numbers  $\{a_n\}$  and  $\{b_n\}$  such that  $\sum_{n=1}^{\infty} a_n = 1 - \sqrt{2}/2$  and  $\sum_{n=1}^{\infty} b_n = \sqrt{2}/2$ .

Define a unital positive linear functional  $F : C(X) \rightarrow \mathbb{R}$  as follows:

$$F(f) = \sum_{n \in \mathbb{N}} a_n f\left(\frac{-1}{n}\right) + \sum_{n \in \mathbb{N}} b_n f\left(\frac{1}{n}\right) \text{ for all } f \in C(X).$$

Let  $D_0 = F(C(X, \mathbb{Z}))$ . Note that, if  $S$  is a clopen subset which does not contain  $[0, 1]$ , then  $F(S) \in \mathbb{Q}$ . If  $S \supset [0, 1]$ , Then

$$F(S) = 1 - F(S_1)$$

for some clopen subset  $S_1 \subset X$  which does not intersect with  $[0, 1]$ . It follows that  $D_0 \subset \mathbb{Q}$ .

This gives a unital positive linear map  $F_* : C(X, \mathbb{Z}) \rightarrow \mathbb{Q}$ . Let  $p \in C(X)$  be a projection whose support  $\Omega$  has a non-empty intersection with  $[0, 1]$ . Since  $\Omega$  is clopen,  $\Omega \supset [0, 1]$ . It follows that there is  $N \geq 1$  such that  $\frac{1}{k} \in \Omega$  for  $|k| \geq N$ . It follows that

$$F(p) \geq \sum_{|k| \geq N} \frac{1}{2^{|k|+1}} > 0.$$

From this one sees that  $F_*$  is strictly positive.

Let  $A$  be a unital simple AF-algebra with

$$(K_0(A), K_0(A), [1_A]) = (\mathbb{Q}, \mathbb{Q}_+, 1).$$

There is an element  $\kappa \in \text{KL}(C(X), A)$  such that

$$\kappa|_{K_0(C(X))} = F_*.$$

Thus  $\kappa(K_0(C(X))_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$ . In other words,  $\kappa \in \text{KL}_e(C, A)^{++}$ .

Suppose that  $\gamma : C_{s.a.} \rightarrow \text{Aff}(T(A)) = \mathbb{R}$  is unital and positive such that

$$\gamma(\check{p})(\tau) = \tau(\kappa([p]))$$

for all projections  $p \in C$  and  $\tau \in T(A)$ . Consider a positive continuous function  $f \in C(X)$  with  $0 \leq f \leq 1$  whose support  $S$  is an open subset of  $(0, 1)$ . Consider projection  $p_n(t) = 0$  if  $t \notin [-1/n, 1 + 1/n] \cap X$  and  $p_n(t) = 1$  if  $t \in [-1/n, 1 + 1/n] \cap X$ . Then

$$f \leq p_n, \quad n = 1, 2, \dots$$

It follows that, for all  $\tau \in T(A)$ ,

$$\gamma(\check{f})(\tau) \leq \gamma(\check{p}_n)(\tau) \tag{e 5.77}$$

$$< \sum_{|k| \geq n} (a_k + b_k) \rightarrow 0 \tag{e 5.78}$$

as  $|n| \rightarrow \infty$ . It follows that

$$\gamma(\check{f})(\tau) = 0 \text{ for all } \tau \in T(A).$$

This shows that  $\gamma$  is not strictly positive.

In particular, there is no unital monomorphism  $\phi : C(X) \rightarrow A$  such that  $[\phi] = \kappa$ .

How about homomorphisms? Suppose that there exists a unital homomorphism  $h : C(X) \rightarrow A$  such that  $[h] = \kappa$ . Let  $f \in C(X)_+$  be so that its support is contained in  $[0, 1]$ . Then, as shown above,  $\tau(h(f)) = 0$  for  $\tau \in T(A)$ . Since  $A$  is simple, this implies that  $h(f) = 0$ . It is then easy to see that

$$\ker h = \{f \in C(X) : f|_{X \setminus (0,1)} = 0\}.$$

Thus  $C/\ker h \cong C(Y)$ , where  $Y = X \setminus (0, 1)$ . Let  $\phi : C(Y) \rightarrow A$  be the unital homomorphism induced by  $h$ . Then  $\phi$  is a monomorphism. Let

$$Y_1 = \{1 + 1/n : n \in \mathbb{N}\} \cup \{1\} \text{ and } Y_2 = \{-1/n : n \in \mathbb{N}\} \cup \{0\}.$$

Then  $Y_1$  and  $Y_2$  are clopen subsets of  $Y$ . Let  $p_i$  be the projection corresponding to  $Y_i$ ,  $i = 1, 2$ . Then

$$\tau(p_1) \geq \sum_{n=1}^{\infty} b_n = 1 - \sqrt{2}/2 \text{ and } \tau(p_2) \geq \sum_{n=1}^{\infty} a_n = \sqrt{2}/2$$

for  $\tau \in T(A)$ . Since  $\tau(p_1) + \tau(p_2) = 1$ , it follows that

$$\tau(p_1) = 1 - \sqrt{2}/2 \text{ and } \tau(p_2) = \sqrt{2}/2.$$

This is impossible since  $K_0(A) = \mathbb{Q}$ .

From this we arrive at the following conclusion:

**Proposition 5.5.** *There are compact metric spaces  $X$  with dimension one, unital simple AF-algebras  $A$  with unique tracial states and  $\kappa \in KL_e(C, A)^{++}$  which has no strictly positive affine map from  $\text{Aff}(T(C(X)))$  to  $\text{Aff}(T(A))$  compatible with  $\kappa$ .*

*Furthermore, there is no unital homomorphism  $\phi : C(X) \rightarrow A$  such that  $[\phi] = \kappa$  in  $KL(C, A)$ .*

**Definition 5.6.** Let  $C$  be a unital AH-algebra which admits a faithful tracial state and let  $A$  be a unital simple  $C^*$ -algebra with  $T(A) \neq \emptyset$ .

Denote by  $KL T(C, A)^{++}$  the set of pairs  $(\kappa, \lambda)$  where  $\kappa \in KL(C, A)^{++}$  with  $\kappa([1_C]) = [1_A]$  and  $\lambda : T(A) \rightarrow T_{\mathbf{f}}(C)$  which is compatible with  $\kappa$ , i.e.,  $\lambda(\tau)(p) = \tau(\kappa([p]))$  for all projections  $p \in M_{\infty}(C)$  and for all  $\tau \in T(A)$ .

Denote by  $\text{Mon}_{au}^e(C, A)$  the set of approximately unitary equivalent classes of unital monomorphisms from  $C$  into  $A$ .

To conclude this note, combining the previous result in ?? (see 3.1) and 5.2, we state the following:

**Theorem 5.7.** *Let  $C$  be a unital AH-algebra which admits a faithful tracial state and let  $A$  be a unital separable simple  $C^*$ -algebra with  $TR(A) = 0$ . Then map*

$$\Lambda : \text{Mon}_{au}^e(C, A) \rightarrow KLT(C, A)^{++}$$

*defined by  $\phi \mapsto ([\phi], \phi_T)$  is bijective.*

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